Indian Statistical Institute
Back Paper Examination
Differential Topology-BMath III

Max Marks: 100 Time: 3 hours

Throughout, X, Y, Z will denote manifolds without boundary unless otherwise stated. All maps are assumed to be smooth. All questions carry equal marks.

(1) Show that the sphere S^n is a manifold by explicitly giving parametrizations. Describe the tangent space to the sphere at any point.
(2) State and prove the local submersion theorem. Show that there does not exist a submersion $f : S^1 \longrightarrow \mathbb{R}$. Give an example of a submersion $g : \mathbb{R} \longrightarrow S^1$.
(3) Show that the determinant function $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is Morse if $n = 2$ and not Morse if $n \geq 3$.
(4) Show that every real valued function on a compact manifold has a critical point. Find the critical points of the function $f : S^1 \longrightarrow \mathbb{R}$ given by $f(x, y) = xy$. Is f Morse?
(5) Show that $X = [0, 1] \times [0, 1]$ is not a manifold with boundary.
(6) State and prove the preimage theorem. Use this to give another proof of Problem 1 above.
(7) Let X be a manifold with boundary. Show that there is no map $g : X \longrightarrow \partial X$ such that g restricted to ∂X is the identity map.
(8) State the Brouwer fixed point theorem. Let A be a $n \times n$ real matrix with all entries non-negative. Show that A must have a non-negative eigenvalue.
(9) Let $f : X \longrightarrow \mathbb{R}^n$ be a map with X compact, connected and of dimension $n - 1$. Let z be a point not in the image of f. Define the winding number $W_2(f, z)$. Let $f : S^1 \longrightarrow \mathbb{C}$ be the map $f(z) = z^3$. Compute $W_2(f, 0)$ and $W_2(f, 2i)$.
(10) State the Borsuk-Ulam theorem. Let $f : S^k \longrightarrow \mathbb{R}^{k+1} - 0$ be an antipode preserving map. Show that f intersects every line through the origin.